Logic in Mathematics and the Sciences

The theme: logic is not the last word

Logic is supposed to provide a secure foundation to mathematics. In mathematics, '*everything is logical and certain*' – such, at least, is the common perception of what mathematics is. There even exists a view that mathematics is *reducible* to logic. In fact, things are not so simple, and logical perfection (or 'completeness') is a goal not attainable in mathematics. I will try to explain to you what this means.

Mathematics deals with objects ('concepts', if you wish) having little reference to the unknown and mysterious world 'out there'. In contrast, the sciences do try to explain the real world, and so nothing can be of ultimate certainty in the natural sciences. Still, science is supposed to follow the *logical method,* and the common perception of science is that it is this logical method that sets science apart from other human pursuits, where the latter involve *extra-logical* factors in varying degrees. Unfortunately, this also is not a complete description of how science inquires into nature.

Mathematics is 'logic plus set theory'

I will talk of mathematics first. Gottlob Frege systematically developed the *logicist* view of mathematics, when he tried to show that the natural numbers can be defined in purely logical terms, without reference to objects of experience. Strictly speaking, logic provides the *language of reasoning,* i.e., a codification of the way all processes of human reasoning and inference are perceived to work, where this language needs a *setting* so that the abstract rules of language can work on concrete 'things' (somewhat like English and Bengali being two different settings in which the same common abstract rules of language take concrete forms). These concrete settings are provided by – *sets.* Thus, a commonly accepted view of mathematics is: *mathematics is logic plus set theory.*

The syntax

As I said, logic means a *language,* that can be described as a collection of *symbols* and precise rules of forming strings out of those symbols – ones referred to as *formulas* and *sentences.*

Languages in mathematical logic come at various *levels* of descriptive power. At the lowest level, one has the *propositional logic,* where symbols for propositions are linked up by means of logical connectives (such as AND, OR, NOT) to form the formulas (*they is a great book* AND *I want to buy it*). Here the propositions are the elementary units where there is no attempt to analyze a proposition into an *object* and a *predicate* (i.e., some kind of a property of the object). Such an analytical apparatus is provided by a higher level language, namely, the *predicate logic* (or, more precisely, the *first order* predicate logic). For instance, the object 'book' belongs to the class 'great' (the predicate). Predicate logic is capable of talking about objects by means of predicates.

Incidentally, the second order predicate logic is capable of a still higher level of description – one can talk *about* predicates ('she is a kind person, and *kindness* is a *virtue*'). The commonly accepted view is that mathematics can be mostly described by means of first order predicate logic. It was the great contribution of Frege that he developed the idea of predicate logic.

A language in (first order) predicate logic consists of logical symbols (including the *quantifier* symbols $\forall$ and $\exists$), a set of symbols for *variables*, a set of *function* symbols (ex: $S(x) = x + 1$; the constant symbols are a special case), and a set of *relation* (or *predicate*) symbols (ex: $x < y$), because these are the types of things that are mostly needed in mathematics. A language is realized (or *interpreted*) by means of a *structure*, namely, a *set*, whose elements constitute the 'universe' of discourse, where the set is endowed with functions and relations that constitute the interpretations of the language symbols (ex: the set of *natural numbers* constituting a structure for the *language of number theory*; the set of real numbers provides another structure in the same language).

The symbols in a language are put together to form *formulas* (meaningful objects) according to a definite set of rules (ex: ' $x \leq x^2$ ', but not ' $<x^2$ ') and, at the next level, *sentences (*assertions that can be either *true* or *false;* ex: $(\forall x)\ x < x^2$; here the variable x is a *bound* one*).*

The semantics

The rules of formation of formulas and sentences define the *syntax* of the language (much like the grammar in English or Bengali language) where only the *forms* of the strings of symbols are relevant. However, these strings also have to have some *meaning* so that one can say whether a given sentence is true or false. This relates to the *semantics* of the language, involving an interpretation in terms of concrete objects (elements of some specified set), and this is where a *structure* comes in. A language can have more than one structures in terms of which it can be interpreted. For instance the language L^* of natural numbers can be interpreted in the set of the natural numbers, or alternatively in terms of the set of the rational numbers, or even in the set of real numbers.

The assignment of meaning (truth or falsity) to the formulas and sentences in a language in predicate logic involves detailed (and intricate) considerations. As regards formulas, one speaks of their *validity* in any given interpretation. Since the language accommodates variables, functions, and relations (predicates), the question of the validity depends on the assignment of *values* to the variables appearing in the functions and relations. Thus a formula such as $x < x + 1$ is a valid one since it is true for all assignments of values to the variable x, while $x < 2$ is not a valid formula since it is true for only some particular assignments. When one speaks of the validity of a formula in a *language*, however, one requires that the formula be valid in *all* the possible interpretations of the language (analogous to the phrase *a young lady,* which is a valid phrase, having different forms in English and Bengali). In the case of a sentence, the corresponding concept is that of *truth*, where again, the truth (or falsity) is to be same in all the interpretations (analogous to, say, *this is a great book*).

Logical implication and proof

Having talked of syntax and semantics in a language, it remains to introduce the other big concepts in mathematical logic, namely that of *implication* (or logical *consequence*) and *proof* (or *deduction*). The idea of implication originates from that of truth of a sentence. If it so happens that whenever a set of sentences A is true, some other sentence B is also true (in all interpretations, in all value assignments), one says that the sentences in A *imply* B. As in the case of the other concepts, this captures in a precise manner our common sense notion of implication. The notion of *proof* is a related but distinct one. Given a set of sentences A and another sentence B, how can we arrive at B from A in a step-by-step procedure, i.e., in other words, how can we *infer* B from A (like, starting from a set of axioms, we arrive at a theorem)? *Proof* is the ultimate thing for a mathematician. The common man's perception that mathematics is utterly logical rests on the fact that whatever the mathematician says, she backs it up with a proof.

Soundness and completeness

As in the case of the other concepts, mathematical logic gives you a precise definition of the concept of a proof as well. The fundamental requirement that a proof system (or a *deductive* system) is to satisfy is that it must not produce a proof of a false statement (i.e., a statement B whose negation is implied by the axioms A). This is known as *soundness* of the deductive system. It turns out that a reasonably defined proof system in predicate logic is indeed a sound one.

The converse to soundness is no less crucial and is referred to as *completeness* : *if the truth of a sentence B is implied by that of a set of axioms A, then one should be able to have a proof of B from A.* In other words, one should be able to *determine* the truth of a sentence (conditional on the truth of the axiom set A) by some precisely defined procedure.

The point here is that the truth or falsity of a sentences in a logic is determined by the rules of the language independently of the mathematician and is not known *a prori* to her: she has to *discover* it, much like an explorer setting out to discover a canyon in a difficult terrain. The canyon may or may not exist (it may be a mythical one), but even if it exists, that existence does not automatically imply a *path* from the explorer's camp leading up to the canyon. *If* the path does happen to be there, only then the explorer's efforts to discover the canyon will lead to a successful expedition (a good explorer is adept at finding the right path from among a set of paths, most of which will lead her astray, while a bad explorer may fail to chart out the path, somewhat like a not-so-bright mathematician failing to prove a theorem even when the proof exists; this brings up another interesting question, of which more later).

Kurt Gödel established the completeness theorem in predicate logic that tells us that true sentences are indeed provable, thereby confirming the common sense perception that mathematics is the ultimate in logic (this must have been psychologically satisfying to Gödel who believed in perfection of things).

Incompleteness: the limits of logic

This, however, is not the end of the story. Because Gödel then went on to establish the two famous *incompleteness* theorems bearing his name. These relate to the *language of number theory* ( ). The first incompleteness theorem refers to an axiomatization (i.e., a choice of axioms in the natural interpretation, where the universe of discourse is the set of natural numbers) that can be said to be a *minimal* one in that it does not include the *principle of mathematical induction* (recall the way you derived the binomial theorem by invoking this principle). The axiom set featuring in the first incompleteness theorem is a *consistent* and *recursive* one (a couple of desirable characteristics for an axiom system), and the theorem states that this system *cannot prove every true sentence about the set of natural numbers*. Notice that we are not making a statement about true sentences in the *language* of number theory (but only in a particular interpretation) because then any true statement would be provable by the completeness theorem.

The second incompleteness theorem makes use of the *Peano axiomatic system* of arithmetic which is stronger than the minimal choice mentioned above and yet cannot prove all true statements about natural numbers. In particular, it cannot prove its own *consistency* (a system is *inconsistent* if it proves a sentence A and, at the same time, the negation of A; if not, it is said to be consistent; a system has to be either consistent or inconsistent; what is remarkable is that even if Peano arithmetic is consistent, that fact cannot be proved within its folds).

Gödel established the two incompleteness theorems by making use of an ingenious *numbering* of all possible formulas in the language of number theory, referred to as *Gödel numbering,* that made possible the construction of formulas that made statements *about themselves.* In particular, he constructed a particular self-referencing sentence (the so-called Gödel sentence) that, by its very construction, was true but not provable (Gödel himself must have been unhappy with these two theorems of his; he was a troubled person and did not easily reconcile to contrariness). Initially, some people were dismissive about the incompleteness theorems, thinking that the Gödel type sentences were contrived and kinky ones (like the statement in the *liar paradox*) and are not relevant for 'real' maths. But the second incompleteness theorem and a number of subsequent developments established that incompleteness is a basic feature of arithmetic (and indeed of all of mathematics) that cannot be wished away or dismissed as innocuous.

The incompleteness theorems gave a big jolt to the world of mathematics, telling people that mathematics was not all that neatly and logically tied up. In particular, these put a stop to a grand program initiated by David Hilbert where he demanded a demonstration to the effect that there exists a finitary proof of every (correct) theorem of mathematics.

The guessing game

With this brief introduction to the 'lack of logic' in mathematics, I will now turn to another greatly interesting problem I hinted at above while giving you the analogy of the explorer. As the explorer, sitting at the base camp, plans her strategy, she has before her a choice of several paths leading away from the camp and, for each of these, again a choice from among several alternatives as that path branches out after some distance. Overall, it may turn out (as it does turn out in the case of a deductive system) that there is an incredibly complex maze of paths and branches from which she will have to choose one particular course most likely to lead her to the canyon (*if,* that is, *the canyon exists*; see the figure below, in which A denotes the base camp and B the canyon; a few branchings are shown; the paths through C and D lead to the canyon from E and F).

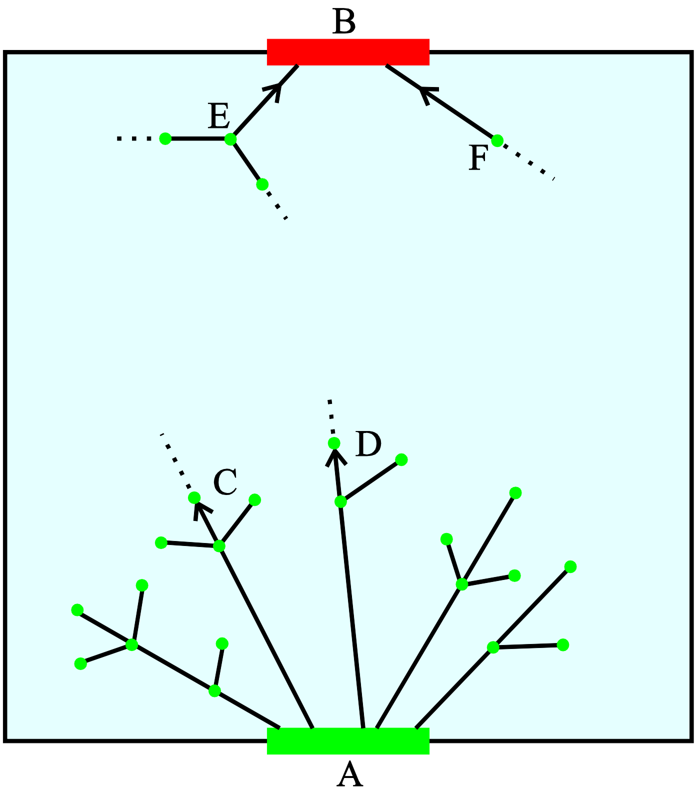

Fig.: Exploring the canyon

Analogously, the mathematician, guessing at a true sentence (a theorem) starts from her axiom set and has to find a correct path (a proof; there may possibly be a number of alternative proofs) leading to the theorem. As the incompleteness theorem tells us, it is even possible that there does not exist a proof. Supposing that a proof exists, how does she guess which of the innumerable alternatives to follow? Does she try out all the possible alternatives one by one like a robot, discarding a choice when it proves to be a wrong one (how does she *know* that it is a wrong choice?) - obviously not. It seems that the mathematician is having to play an incredibly absurd guessing game, which is precisely why she is immortalized on proving a theorem no one else has succeeded in proving. Evidently, she is not playing a game of *blind* guess – but there is no doubt that it *is* a guessing game. Once she is convinced that she has hit upon the correct guess, she proceeds to set up a logically watertight demonstration of the proof of the theorem at hand.

What is a guess and how does one successfully make a guess where others continue to remain puzzled? This question is dismissed in mathematics and in the natural sciences as one belonging to the domain of *psychology.* What people require of the mathematician or the scientist is a logical demonstration of the correctness of what she has successfully guessed at. It is as if she is all the while having to do deep water diving to collect precious stones from the sea-bed and all that people are interested in is to be convinced by means of intricate tests that the stones are indeed genuine ones, not caring to bother about the remarkable processes of diving and collection.

It has been told of Ramanujan, one of the all time great mathematicians, that he did not really care about formal proofs, despite having correctly guessed at an astounding collection of true statements in number theory. Similar anecdotes have been told of Gauss, the 'prince' of mathematicians, for whom theorems sprang up from within his mind at such rapid succession that he did not have the time for writing down the formal proofs and getting those published or otherwise made known.

Guessing means a choice from among alternatives, without sufficient explicit *reason* for making the choice (otherwise anybody with a sufficient reasoning ability could have made the guess). In other words, guessing is more or less an *extra-logical* activity where the apparatus of formal logic cannot quite give us a clue as to how the guessing game is played out. It is this act of *guessing without sufficient reason* that is common to the way mathematics and the natural sciences are actually practiced. Strictly speaking, it is this same guessing game that is played out in every human activity – mathematical, scientific, or not. What distinguishes mathematics and the sciences from other areas of human activity is the way the guesses are *justified*, where the process of justification makes use of the apparatus of logic – specifically, of the notions of logical implication and deduction.

The context of discovery and the context of justification

In other words, one has to distinguish between the context of discovery and the context of justification (though, in real life, the two are blended together, like everything else in this world). As regards justification, mathematics and the sciences are indeed more logical (or more *objective*) as compared to, say, selecting a cricket team from among probables, or economic forcasting. Between mathematics and the natural sciences, justification is rigorous and precise in the former, while in the sciences there remains a much greater degree of uncertainty and complexity since one has to play out the guessing game all the way.

Induction: the 'scandal of philosophy'

The act of choosing, without sufficient reason, a hypothesis from among alternatives is basically the same as the much-discussed process of *induction,* where one generalizes from a number of particular instances, and where there is, once again, insufficient reason for the generalization. Ever since David Hume posed the problem of induction in clear terms, attempts at understanding induction by philosophers and scientists have mostly gone unrewarded, as a result of which induction has been referred to as the 'scandal of philosophy'.

Karl Popper, the noted philosopher of science, wanted to come up with a clear-cut criterion demarcating science from the pseudo-sciences, and refused to compare the two in terms of the context of discovery, calling the latter a psychological process that does not lend itself to an objective analysis. According to him, science distinguishes itself by the process of justification where there takes place a continual referral to results of *observation and experimentation* (according to him, a scientific theory cannot really be justified, but can be *falsified*). Other analysts have been less dismissive of the context of discovery, and have inquired into the details of the guessing process that gives the mathematician the idea of a new theorem or the scientist a new hypothesis. In particular, the cognitive sciences have, in recent times, looked into the process of induction and other associated 'learning' and 'decision making' processes in concrete terms where controlled experimentation and computer programs have played a significant role.

One thing that emerges is that it is really not easy to dissociate the *logical* from the *extra-logical* in the process of scientific inquiry. For instance, guessing and interpreting (both having inductive components) are involved at every stage of scientific activity, even down to the level of fitting the results of experiments to a hypothesis or a theory.

What, in concrete terms, is involved in the extra-logical process of induction and hypothesis building? One important component of this process involves what is referred to as *heuristics,* where a heuristic is a tentative and temporary generalization at a lower level, playing the role of a stepping-stone toward a bigger generalization (example (in chess): *grab the queen;* example (everyday life): *look, son, this is a circle; and that other figure, that is a circle too; now you know what a circle is*).

Extra-logical factors in guessing: tacit knowledge

It seems possible that very many things get involved in the extra-logical process of building up heuristics at various levels, including personal beliefs, and culturally acquired mind-sets. And, what is more, all these possibly operate at a substratum of the human mind, forming the *tacit* component of the cognitive-inductive process. It was Michael Polanyi, the noted physical chemist and philosopher of science who first developed the notion of *tacit knowledge* – knowledge that one possesses but is incapable of expressing cogently. A considerable amount of experimental and theoretical work has been done in recent decades on this area of human cognition. Maybe, in days to come, scientists and philosophers will seriously look into the psychological process of guessing and hypothesis building, and will tell us more on the role that extra-logical factors play in the scientists' inquiry into nature.

Church's thesis

If you are still with me, I add here an optional topic for those interested in computer science. Let us confine ourselves to the set of natural numbers (including zero) and sets of n-tuples (n=1,2,3...) of numbers, that can act as arguments of functions. We start with the (intuitive) idea of *effectively computable* functions, i.e., those whose values for any given arguments can be calculated by step-by-step finite procedures (this is basically what a computer does). One can then define effectively computable *sets* and *relations* as well. The question that arises here, can this idea of an effective computation be made precise? Note that the question itself is not very precise since the idea that we want to define precisely, that of effective computation, is not quite a precise one. However, there exist a number of precisely defined notions that come very close to our idea of effective computation. One is the notion of *recursive* functions (and recursive sets and relations). The set of recursive functions can be defined in logical terms using the so-called notion of *representability* (Gödel) or else in terms of a set of operations on certain *basic* functions, where the latter idea is closer to the notion of effective computation. Another precisely defined notion is that of *Turing computation* (developed by making use of *Turing machines*), while other similar notions are also there. The *Church-Turing thesis* (not a theorem!) asserts that all these different notions are equivalent, thereby providing a very strong basis for the intuitive idea of effective computation.

The idea of effective computability can be invoked to define that of *decidability,* and an alternative approach to the incompleteness theorems can be given in terms of *undecidability* of a *theory of arithmetic.*

I have to add here that the notion of computability does not, in real life, set the limits of deductive logic. In other words, even before the horizon set by incompleteness or computability is reached, our ability to seek mechanical solutions to challenging problems (to problems in diverse areas of natural science, for instance) comes up against the obstacle of *complexity.* A problem may be, in principle, computable but, at the same time, too complex to actually obtain a solution to it. For instance, it is most likely that an *exponentially large* time may be needed by a computer program to find a solution to an arbitrarily specified *traveling salesman* problem or to work out the *quantum correlation* existing between two systems of arbitrary size.

Learning algorithms and human cognition: beyond 'logic'

This talk focuses on the theme of 'limits of deductive logic' in mathematics and the sciences. But deductive logic is not the last word on the reasoning powers of the human mind.

Even if the barriers set by complexity are ignored (but they *can't be!*), deductive logic comes up against the impregnable wall of incompleteness and computability. There is no escaping the fact that there exist *non-computable* functions just as there exist unprovable true statements in arithmetic (indeed, non-computable functions are more numerous that the computable ones). How can non-computable functions be evaluated? Or, how can unprovable true statements (the Goldbach conjecture or the twin prime conjecture are *probable* examples) be arrived at? Or, more importantly, how can a simple and meaningful insight be obtained in a computationally complex problem? Can the human mind work its way into this vast incomprehensible world in spite of the limits set by complexity, undecidability and incompleteness? Happily, there is no end to the cunning of the human mind. It is capable of *guessing, generalizing,* and *deciding,* capable of generating *insights.* There is no way these faculties of the human mind can be explained *within the framework of logic* as we know it. But it is the same mind that never tires of trying to have glimpses into modes of its *own* working.

The past few decades have witnessed the development of *learning algorithms* (neural networks, parallel distributed processing, as also other novel approaches). How do the performance of these algorithms relate to the limits of Turing computation, and capture one or more features of human cognition, including tacit knowledge? These are some of the questions being addressed by investigations in *cognitive science* that take us beyond the realm of 'pure' logic. It is likely that our understanding of the term 'logic' itself will be broadened in days to come.

Suggested reading

1. Christopher C. Leary, *A Friendly Introduction to Mathematical Logic,* Prentice-Hall, Inc (2000).

2. Aidan Feeny and Evan Heit (ed.), *Inductive Reasoning*, Cambridge University Press (2007).

3. Douglas Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books Inc. (1999).

4. William Poundstone, *Labyrinths of Reason: Paradoxes, puzzles and the frailty of knowledge,* Penguin Books (1988).

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